Stochastic Particle Motion

Mean square deviation (MSD)

\[
\langle (\Delta x)^2 \rangle = \langle (x(t) - x(0))^2 \rangle \sim t^\sigma
\]

Different cases:

\[0 < \sigma < 1\] : Subdiffusion
\[\sigma = 1\] : (Markovian) Diffusion
\[\sigma > 1\] : Superdiffusion

Diffusion coefficient (Kubo-formula)

\[
\kappa_{xx} = \lim_{t \to \infty} \frac{\langle (\Delta x)^2 \rangle}{2t} = \int_0^\infty dt \langle v_x(t)v_x(0) \rangle
\]
Equations of Motion

Parallel diffusion:

\[ \dot{\mu} = \frac{\Omega}{v} \left( v_x \frac{\delta B_y}{B_0} - v_y \frac{\delta B_x}{B_0} \right) \]

\[ \rightarrow D_{\mu\mu}(\mu) = \int_0^\infty dt \left\langle \dot{\mu}(t) \dot{\mu}(0) \right\rangle \]

\[ \rightarrow \kappa_\parallel = \frac{v^2}{8} \int_{-1}^1 d\mu \frac{(1 - \mu^2)^2}{D_{\mu\mu}(\mu)} \]

Perpendicular diffusion:

\[ v_x = v_z \frac{\delta B_x}{B_0} \]

\[ \rightarrow \kappa_\perp = \int_0^\infty dt \left\langle v_x(t) v_x(0) \right\rangle \]
A Review: Theories for Diffusion of Charged Particles

Turbulence and Transport Theory

In Cosmic Ray diffusion theory terms of the form

$$\langle \delta B_i(\vec{x}(t))\delta B_j(\vec{x}(0)) \rangle$$

occur!

Employ a Fourier Transformation

$$\delta B_i(\vec{x}(t)) = \int d^3 k \delta B_i(\vec{k}, t) e^{i\vec{k} \cdot \vec{x}(t)}$$

to find

$$P_{ij}(\vec{k}, t) = \langle \delta B_i(\vec{k}, t)\delta B_j^*(\vec{k}, 0) \rangle.$$

**Problem I:** What are the magnetic fields/correlation functions described by $P_{ij}(\vec{k}, t)$? ⇒ see first talk!

**Problem II:** How can we replace the particle trajectory $\vec{x}(t)$?
Principle of the quasilinear theory (QLT, Jokipii 1966):

replace $\vec{x}$ by the unperturbed trajectory

$\Rightarrow$ QLT is a first-order perturbation theory!
Quasilinear Theory

The resonance functions $R_{\pm}$ describe the interaction between plasma and particle. Example for a quasilinear result:

\[
D_{\mu \mu} = \frac{2\pi v^2 (1 - \mu^2)}{B_0^2 R_L^2} \int_0^\infty dk_\parallel g_{\text{slab}}(k_\parallel) \left[ R_-(k_\parallel) + R_+(k_\parallel) \right]
\]

(Particles, Fields, Interaction)

Resonance function of QLT

\[
R_{\pm}^{(QLT)} = \pi \delta \left( k_\parallel v_\parallel \pm \Omega \right)
\]

⇒ only interaction for

\[
\frac{v_\parallel}{\Omega} \equiv \mu R_L = \frac{1}{k_\parallel}
\]

(Gyro-resonance)
The parallel mean free path: simulations and quasilinear theory.
The perpendicular mean free path: *simulations*. In this case we find by employing QLT $\lambda_\perp/\lambda_\parallel = \infty$. 
Nonlinear Particle Diffusion Theory

Weakly Nonlinear Theory (WNLT) developed by Shalchi et al. (2004):

Idea: nonlinearity due to resonance broadening

\[ R_{\pm}^{(QLT)} = \pi \delta (k \parallel v \parallel \pm \Omega) \]

\[ \rightarrow R_{\pm}^{(WNLT)} = \frac{\alpha}{\alpha^2 + (k \parallel v \parallel \pm \Omega)^2} \]

with \( \alpha = \alpha (\kappa_i) \).
The parallel mean free path: simulations, quasilinear theory, and nonlinear theory of Shalchi et al. 2004.
The parallel mean free path: simulations, quasilinear theory, and nonlinear theory of Shalchi et al. 2004.
The perpendicular mean free path: simulations and nonlinear theory of Shalchi et al. 2004. The QLT-result is $\lambda_\perp / \lambda_\parallel = \infty$. 
Perpendicular Diffusion: Nonlinear Theory and Simulations

The perpendicular mean free path: simulations and nonlinear theory of Shalchi et al. 2004. The QLT-result is $\lambda_\perp/\lambda_\parallel = \infty$. 
Application I - Particles in the Solar System

The perpendicular mean free path: theoretical results from Shalchi et al. (2006) for electrons and protons in comparison with different observations: Jovian electrons, Ulysses measurements of Galactic protons, and the value suggested by Palmer (1982).
Transport of cosmic rays in the Galaxy:

Interaction between cosmic rays and the interstellar medium

⇒ Primary cosmic particles decay in lighter nuclei
⇒ Secondary cosmic rays are generated

⇒ The measured decrease of the abundance ratio of secondary to primary cosmic ray nuclei as $B/C$ and $N/O$ at kinetic energies above 1 GeV per nucleon allows the determination of the life time of the particles in the Galaxy
The measurements indicate that (see, e.g., Swordy et al. 1990):

\[ v \tau \sim R^{-a}, \quad a = 0.6 \pm 0.1 \]

\( \tau \) = life time of the particles in the Galaxy

For diffusive transport:

\[ \kappa_\parallel = \frac{< (\Delta z)^2 >}{2t} \sim \frac{L^2}{2\tau} \sim v \lambda_\parallel \]

\( L \) = thickness of the Galactic disk

and, therefore,

\[ \lambda_\parallel \sim \frac{1}{v \tau} \sim R^a \]
The measurements indicate that: \( \lambda_{\parallel} \sim R^{0.6} \sim E^{0.6} \) (Swordy et al. 1990)

The parallel mean free path in the interstellar medium: quasilinear result in comparison with a \( R^{0.6} \) – fit and the nonlinear result from Shalchi & Schlickeiser (2005).
Application II - Transport of Cosmic Rays in the Galaxy

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Application II - Transport of Cosmic Rays in the Galaxy

Nonlinear scattering of low energy cosmic rays in the Galaxy (see Shalchi & Büsching, 2010, submitted)

⇒ Break in the rigidity dependence at $R \approx 3\text{GV}$!
**Summery & Outlook**

**Summery:**
- To understand the motion of charged cosmic rays we need an improved understanding of MHD-turbulence
- Nonlinear effects are essential for computing charged particles orbits
- Nonlinear diffusion theories are relevant for describing cosmic ray propagation and acceleration in the solar system, the ISM, ...

**Ongoing and Future work:**
- Improved (phenomenological) models for turbulence (Maltese cross, general wave spectrum, dynamical turbulence)
- Further improvement of the (nonlinear) scattering theory and further applications
- Computer-simulations for realistic turbulence models
A Review: Theories for Diffusion of Charged Particles

For a review see Book: